Recitation 8. May 4

Focus: positive definite/semidefinite matrices, singular value decomposition, pseudo-inverse

If all the eigenvalues (or equivalently, the pivots) of a symmetric matrix S are positive/non-negative, then S is called:

$$\boxed{\textbf{positive definite/semidefinite}}$$

We have:

$$\begin{array}{lcl} S \text{ positive definite} & \Leftrightarrow & \boldsymbol{v}^T S \boldsymbol{v} > 0 \\ S \text{ positive semidefinite} & \Leftrightarrow & \boldsymbol{v}^T S \boldsymbol{v} \geq 0 \end{array}$$

for any vector $\boldsymbol{v} \neq 0$. The quantity $\boldsymbol{v}^T S \boldsymbol{v}$ is called the $\boxed{\textbf{energy}}$ of $\boldsymbol{v}$.

The $\boxed{\textbf{Singular Value Decomposition}}$ (SVD) of a matrix A is a way of writing it as:

$$\boxed{A = U \Sigma V^T}$$

where U and V are orthogonal matrices, and Σ is diagonal. If we let:

$$U = \left[\ \boldsymbol{u}_1 \,\middle|\, \dots \,\middle|\, \boldsymbol{u}_m\ \right] \quad \text{and} \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & \dots \\ 0 & \ddots & 0 & 0 & \dots \\ 0 & 0 & \sigma_r & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad V = \left[\ \boldsymbol{v}_1 \,\middle|\, \dots \,\middle|\, \boldsymbol{v}_n\ \right]$$

then the SVD is a way of writing A as a sum of rank 1 matrices:

$$\boxed{A = \sum_{i=1}^{r} \boldsymbol{u}_i \sigma_i \boldsymbol{v}_i^T}$$

You may compute the SVD by letting:

- the **right singular vectors** $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ be the eigenvectors of $A^T A$
- the **left singular vectors** $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m$ be the eigenvectors of AA^T
- the **singular values** $\sigma_1, \dots, \sigma_r$ be the square roots of the non-zero eigenvalues of $A^T A$ or AA^T

The $\boxed{\textbf{pseudo-inverse}}$ of A is defined by:

$$\boxed{A^+ = V \Sigma^+ U^T}$$

where Σ^+ has diagonal entries $\frac{1}{\sigma_i}$ instead of σ_i. It is useful because:

$$\boxed{\text{the closest } A\boldsymbol{v} \text{ can be to a vector } \boldsymbol{b} \text{ is achieved for } \boldsymbol{v} = A^+ \boldsymbol{b}}$$

Moreover:

$$\begin{array}{l} AA^+ \text{ is the projection matrix onto } C(A) \\ A^+ A \text{ is the projection matrix onto } C(A^T) \end{array}$$

Also, the SVD allows us to define the $\boxed{\textbf{polar decomposition}}$ of $A = U \Sigma V^T$ as:

$$\boxed{A = QS}$$

where $Q = UV^T$ is orthogonal, and $S = V \Sigma V^T$ is symmetric.

1. Consider the matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

- Say which of them is positive definite, positive semidefinite, or neither.
- Write down the energy of an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with respect to either of these matrices. Setting the energy equal to 1 gives rise to a conic. What kind of conic is it, in each of the three cases?

Solution:

2. Consider the matrix

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

- Compute the Singular Value Decomposition of A.
- Compute the pseudo-inverse A^{+}. Then compute the inverse A^{-1} by another method. How do they compare?

Solution:

3.

- Express the function:

$$\frac{3x^2 + 2xy + 3y^2}{x^2 + y^2} \tag{1}$$

  in the form $\frac{\boldsymbol{v}^T S \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{v}}$, where $\boldsymbol{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and S is a certain symmetric matrix that you are free to choose. Compute the maximum of (1) in terms of the eigenvalues of S. For what values of (x, y) is the maximum achieved?

- Find the maximum of the function:

$$\sqrt{\frac{(x + 4y)^2}{x^2 + y^2}}$$

  by expressing it in the form $\frac{\|A\boldsymbol{v}\|}{\|\boldsymbol{v}\|}$ for a suitable matrix A, and then invoking the singular values of A.

Solution: